B Online Appendix for 'How to Sell Hard Information' by S. Nageeb Ali, Nima Haghpanah, Xiao Lin, and Ron Siegel

B.1 Proof of Proposition 1 on p. 16

Proof. Consider a test-fee structure (G, ϕ). If (w-P) is violated, then Lemma 1 shows that there is an equilibrium with zero revenue and the proposition follows. Suppose that (w-P) holds. Using integration by parts, we can rewrite (w-P) as $\phi_t \leq \int_{\mu+\phi_d}^{\overline{\theta}} [1 - G(s)]ds$. This expression implies that any testing fee that satisfies (w-P) is at most the area above the score distribution G from $\mu + \phi_d$ to $\overline{\theta}$, shaded dark in Figure 8. The revenue from disclosure is at most $\phi_d \Pr[s \geq \underline{\theta} + \phi_d]$, shaded light in Figure 8. This is because in any equilibrium, a score strictly less than $\underline{\theta} + \phi_d$ strictly prefers to conceal. So the total revenue is at most the shaded area above G. Since G is a mean-preserving contraction of the prior distribution,

$$\mu = \int_{\underline{\theta}}^{\overline{\theta}} s dG(s) = \int_{\underline{\theta}}^{\overline{\theta}} [1 - G(s)]ds + \overline{\theta}$$

and therefore the area above G is equal to $R_F = \mu - \overline{\theta}$.

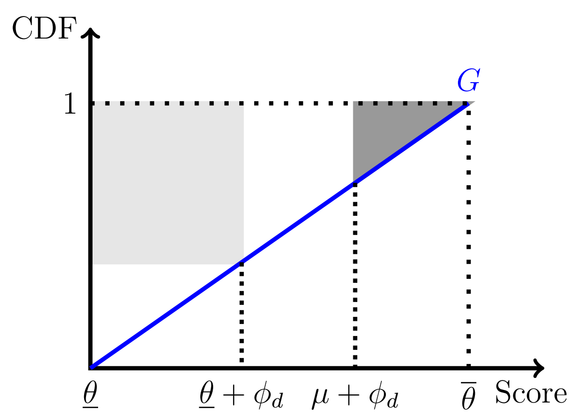

Figure 8: The revenue from the testing fee is shaded dark. The revenue from the disclosure fee is at most the area shaded light.

Now suppose that there exists an equilibrium with a revenue of $R_F - \varepsilon$. Since the revenue is at most the shaded area above G and the total area above G is R_F, the unshaded area above G is at most ε. In particular, the area above G from $\underline{\theta} + \phi_d$ to $\mu + \phi_d$ is at most ε. Since G is monotone,

$$\phi_t \leq \int_{\mu+\phi_d}^{\overline{\theta}} [1 - G(s)]ds \leq \left(\frac{\overline{\theta} - (\mu + \phi_d)}{\mu - \underline{\theta}}\right) \int_{\underline{\theta}+\phi_d}^{\mu+\phi_d} [1 - G(s)]ds \leq \left(\frac{\overline{\theta} - \mu}{\mu - \underline{\theta}}\right) \varepsilon.$$

Therefore, as ε goes to zero, the revenue from the testing fee goes to zero as well. Thus, to complete the proof we only need to show that the revenue from the disclosure fee also goes to zero.

For ε small enough so that $\mu + \phi_d \geq \underline{\theta} + \phi_d + \sqrt{\varepsilon}$, we have

$$\varepsilon \geq \int_{\underline{\theta}+\phi_d}^{\mu+\phi_d} [1 - G(s)]ds \geq \int_{\underline{\theta}+\phi_d}^{\underline{\theta}+\phi_d+\sqrt{\varepsilon}} [1 - G(s)]ds \geq \sqrt{\varepsilon}(1 - G(\underline{\theta} + \phi_d + \sqrt{\varepsilon})),$$

where the third inequality follows since G is monotone. That is, the probability that the score is more than $\tau \equiv \underline{\theta} + \phi_d + \sqrt{\varepsilon}$ is at most $\sqrt{\varepsilon}$. Thus, if there exists an equilibrium threshold above τ, the disclosure probability in that equilibrium is at most $\sqrt{\varepsilon}$. To show that there exists an equilibrium threshold above τ, we apply Lemma 3 by showing that $E[s|s \leq \tau] > \tau - \phi_d$.

The expectation of G can be written as

$$\mu = G(\tau)E[s|s \leq \tau] + (1 - G(\tau))E[s|s > \tau] \leq G(\tau)E[s|s \leq \tau] + (1 - G(\tau))\overline{\theta}.$$

Rearranging terms yields

$$\overline{\theta} - E[s|s \leq \tau] \leq \frac{\overline{\theta} - \mu}{G(\tau)} \leq \frac{\overline{\theta} - \mu}{1 - \sqrt{\varepsilon}}.$$

Therefore, since $\tau = \underline{\theta} + \phi_d + \sqrt{\varepsilon}$ and $\underline{\theta} < \mu$, for $\varepsilon < (\frac{\mu - \underline{\theta}}{1 + \overline{\theta}})^2$ we have

$$\tau - \phi_d = \underline{\theta} + \sqrt{\varepsilon} < \frac{\mu - \overline{\theta}\sqrt{\varepsilon}}{1 - \sqrt{\varepsilon}} = \overline{\theta} - \frac{\overline{\theta} - \mu}{1 - \sqrt{\varepsilon}} \leq E[s|s \leq \tau],$$

and therefore by Lemma 3 there is an equilibrium threshold higher than τ.

To complete the proof, recall that $\phi_d < \overline{\theta} - \mu$, so the revenue from the disclosure fee is no more than $\sqrt{\varepsilon}(\overline{\theta} - \mu)$. Thus the total revenue is at most $(\frac{\overline{\theta} - \mu}{\mu - \underline{\theta}})\varepsilon + \sqrt{\varepsilon}(\overline{\theta} - \mu)$. □

B.2 Proof of Proposition 4 on p. 25

We first simplify the step-exponential-step distribution identified in Section 5 when the testing fee is zero.

Lemma 12. *Suppose that the prior distribution is log-concave and consider an optimal step-exponential-step test-fee structure (G, ϕ) defined by (5), (6), and (7). If $\phi_t = 0$, then $\tau_2 = \mu + \phi_d < \overline{\theta}$ and $G(\tau_2) = 1$. Additionally, there exist $\delta, \sigma > 0$ such that $\delta + \int_{\underline{\theta}}^{\tau} G(s)ds \leq \int_{\underline{\theta}}^{\tau} F(s)ds$ for any $\tau \in [\tau_1 - \sigma, \tau_2]$.*

Proof. For the optimal testing fee to be zero, the area above G from $\mu + \phi_d$ to $\overline{\theta}$ must be zero, and hence $G(\mu + \phi_d) = 1$. From Lemma 11, $G(\tau_2) = 1$.

The mean-preserving constraints which require that G is $\overline{\theta}-\mu$ can be written as

$$\begin{aligned}\overline{\theta}-\mu &= \int_{\underline{\theta}}^{\tau_1} G(s)ds + \int_{\tau_1}^{\tau_2} \kappa e^{\frac{s-\tau_1}{\tau_1-\tau_0}} ds + (\overline{\theta}-\tau_2)\\ &= \kappa(\tau_1-\tau_0) + (\tau_1-\tau_0)\kappa e^{\frac{s-\tau_1}{\tau_1-\tau_0}}\Big|_{\tau_1}^{\tau_2} + (\overline{\theta}-\tau_2)\\ &= (\tau_1-\tau_0) + \overline{\theta}-\tau_2,\end{aligned}$$

where the third inequality follows because $\kappa = e^{\frac{\tau_1-\tau_2}{\tau_1-\tau_0}}$ is pinned down by the mean-preserving contraction conditions. We must therefore have $\tau_2 = \mu + (\tau_1-\tau_0) = \mu + \phi_d$.

We now show that the mean-preserving contraction constraints must be slack on interval $[\tau_1, \tau_2]$. Notice that log-concavity implies continuity in the interior, which is used in the later arguments.

We first claim that $F(\tau_1) \geq G(\tau_1) = e^{-\frac{\mu-\tau_0}{\tau_1-\tau_0}}$. Suppose that $F(\tau_1) < G(\tau_1)$. Then $F(x) < G(x)$ for any $x \in [\tau_0, \tau_1)$, which implies the constraint must be slack on the interval $[\tau_0, \tau_1)$ because $\int_{\underline{\theta}}^{x} F(s)ds = \int_{\underline{\theta}}^{\tau_1} F(s)ds - \int_{x}^{\tau_1} F(s)ds > \int_{\underline{\theta}}^{\tau_1} G(s)ds - \int_{x}^{\tau_1} G(s)ds = \int_{\underline{\theta}}^{x} G(s)ds$. Since the integrals are continuous, the constraint is also slack at τ_1. So we can construct a new distribution parameterized by $\tau_1' = \tau_1 - \varepsilon$ and $\tau_0' = \tau_0 - \varepsilon$, the distribution on $[\tau_1, \overline{\theta}]$ does not change so all the constraints on $[\tau_1, \overline{\theta}]$ are still satisfied. Also since the original constraints on $[\underline{\theta}, \tau_1]$ are slack, they are still satisfied for small ε. By charging the same disclosure fee $\phi_d = \tau_1 - \tau_0$, this new distribution induces a higher disclosure probability, which contradicts G being optimal.

If $F(\tau_1) = G(\tau_1)$, the same argument goes through if the constructed distribution does not violate the mean-preserving constraints. If the constraint is slack at τ_1, then all the constraints at points lower than τ_1 are slack, so the constructed distribution is still a profitable deviation. If the constraint binds at τ_1, the right derivative of F at τ_1 must be greater than the right derivative of G. Also, the left derivative of F must be greater than that of G by log-concavity. So the local change of the distribution doesn't violate any constraints, which leads to a profitable deviation.

We must therefore have $F(\tau_1) > G(\tau_1)$ and the constraint is slack at τ_1 due to the continuity of F. Since the mean-preserving constraint is also slack at τ_2 and the integrals of G and F are continuous, there exists $\delta, \sigma > 0$ such that $\delta + \int_{\underline{\theta}}^{\tau'} G(s)ds \leq \int_{\underline{\theta}}^{\tau'} F(s)ds$ for any $\tau \in [\tau_1 - \sigma, \tau_1]$, and further, $\delta + \int_{\underline{\theta}}^{\tau_2} G(s)ds \leq \int_{\underline{\theta}}^{\tau_2} F(s)ds$. Moreover, from the log-concavity of F and the facts that $F(\tau_1) > G(\tau_1)$ and $F(\tau_2) < G(\tau_2)$, F crosses the exponential part of G from above exactly once. To see this, notice that $\log(F)$ is concave and $\log(G)$ is linear on $[\tau_1, \tau_2]$, and a concave function can only cross a linear function from above once. Letting x^* denote the intersection point, we have $F(x) > G(x)$ for $x \in [\tau_1, x^*)$ and $F(x) < G(x)$ for $x \in (x^*, \tau_2]$. Now for any

$\tau' \in [x^*, \tau_2]$ we have

$$\delta + \int_{\underline{\theta}}^{\tau'} G(s)ds = \delta + \int_{\underline{\theta}}^{\tau_2} G(s)ds - \int_{\tau'}^{\tau_2} G(s)ds \leq \int_{\underline{\theta}}^{\tau_2} F(s)ds - \int_{\tau'}^{\tau_2} F(s)ds = \int_{\underline{\theta}}^{\tau'} F(s)ds.$$

Similarly, for any $\tau' \in [\tau_1, x^*]$ we have

$$\delta + \int_{\underline{\theta}}^{\tau'} G(s)ds = \delta + \int_{\underline{\theta}}^{\tau_1} G(s)ds + \int_{\tau_1}^{\tau'} G(s)ds \leq \int_{\underline{\theta}}^{\tau_1} F(s)ds - \int_{\tau_1}^{\tau'} F(s)ds = \int_{\underline{\theta}}^{\tau'} F(s)ds.$$

To complete the proof, we show that $\tau_2 < \overline{\theta}$. If $\tau_2 = \overline{\theta}$, then since G is log-linear on $[\tau_1, \tau_2]$ and F log-concave on that interval and the mean-preserving contraction constraints are satisfied, it must be that $G \geq F$ over the interval $[\tau_1, \tau_2]$, and in particular, $G(\tau_1) \geq F(\tau_1)$. But as we argued above, $G(\tau_1) < F(\tau_1)$, which is a contradiction. □

Given Lemma 12 we now prove Proposition 4.

Proof of Proposition 4. By Lemma 11, if there exists a robustly optimal test-fee structure with zero testing fee, then there exists a robustly optimal test-fee structure (G, ϕ) in the step-exponential-step class with zero testing fee. Suppose that G is parameterized by κ and $\tau_0, \ldots, \tau_3$. This test G must satisfy the properties of Lemma 12. Given (G, ϕ), we construct a class of test-fee structures $(G^\varepsilon, \phi^\varepsilon)$ for $\varepsilon \geq 0$ as follows

$$G^\varepsilon(s) = \begin{cases} \kappa & \text{if } s = \tau_0 \\ \kappa e^{(s-\tau_1(\varepsilon))/(\tau_1(\varepsilon)-\tau_0)} & \text{if } s \in [\tau_1(\varepsilon), \tau_2(\varepsilon)] \\ F(s) & \text{if } s \in [\tau_3(\varepsilon), \overline{\theta}], \end{cases}$$

where $\tau_1(\varepsilon) = \tau_1 - \varepsilon$, $\tau_2(\varepsilon)$ is specified below, and $\tau_3(\varepsilon)$ is defined so that G^ε is flat from $\tau_2(\varepsilon)$ to $\tau_3(\varepsilon)$, that is, $\tau_3(\varepsilon) = F^{-1}(\min(\kappa e^{(\tau_2(\varepsilon)-\tau_1(\varepsilon))/(\tau_1(\varepsilon)-\tau_0)}, 1))$. Let $\phi_d(\varepsilon) = \phi_d - \varepsilon$ and define $\phi_t(\varepsilon)$ such that (w-P) holds with equality.

We define $\tau_2(\varepsilon)$ so that the integrals of G^ε and F are the same. To show that such $\tau_2(\varepsilon)$ exists, we show that there is a unique solution x to

$$\begin{aligned} \overline{\theta} - \mu &= \kappa(\tau_1 - \varepsilon - \tau_0) + \int_{\tau_1-\varepsilon}^{x} \kappa e^{\frac{s-\tau_1+\varepsilon}{\tau_1-\varepsilon-\tau_0}} ds + \kappa e^{\frac{x-\tau_1+\varepsilon}{\tau_1-\varepsilon-\tau_0}} (\tau_3(x) - x) + \int_{\tau_3(x)}^{\overline{\theta}} F(s)ds \\ &= \kappa e^{\frac{x-\tau_1+\varepsilon}{\tau_1-\varepsilon-\tau_0}} (\tau_1 - \varepsilon - \tau_0 + \tau_3(x) - x) + \int_{\tau_3(x)}^{\overline{\theta}} F(s)ds, \end{aligned}$$

where $\tau_3(x) = F^{-1}(\min(\kappa e^{(x-\tau_1+\varepsilon)/(\tau_1-\varepsilon-\tau_0)}, 1))$. The derivative of the right hand side with respect to x is

$$\kappa e^{\frac{x-\tau_1-\varepsilon}{\tau_1-\varepsilon-\tau_0}}\left(\frac{\tau_1-\varepsilon-\tau_0+\tau_3-x}{\tau_1-\varepsilon-\tau_0}+\tau_3'(x)-1\right)-\tau_3'(x)F(\tau_3(x)).$$

The derivative $\tau_3'(x)$ exists because F is assumed to have a positive density. Now evaluate this derivative at $\varepsilon=0, x=\tau_2$, where $\tau_3(x)=\overline{\theta}$ and $1=F(\tau_3(x))=\kappa e^{\frac{\tau_2-\tau_1}{\tau_1-\tau_0}}$. So the terms with $\tau_3'(x)$ cancel out and the derivative is

$$\frac{\overline{\theta}-\tau_2}{\tau_1-\tau_0}>0.$$

Therefore, by the implicit function theorem, $\tau_2(\varepsilon)$ is well-defined and $|\tau_2'(\varepsilon)|<\infty$ for small enough ε. It must be that $G^\varepsilon(\tau_2(\varepsilon))\leq 1$, implying that G^ε is a well-defined distribution. Otherwise G^ε is above G which means that the integral of G^ε is larger than the integral of F. Notice that $\tau_2(0)=\tau_2$, which implies that $(G^0,\phi(0))=(G,\phi)$.

We next show that for small enough $\varepsilon>0$, G^ε is a mean-preserving contraction of F. That is, $\int_{\underline{\theta}}^{\tau} G^\varepsilon(s)ds\leq\int_{\underline{\theta}}^{\tau}F(s)ds$ for any $\tau\in[\underline{\theta},\overline{\theta}]$. This inequality follows for all $\tau\leq\tau_1(\varepsilon)$ because G and G^ε are identical below $\tau_1(\varepsilon)$, and G is a mean-preserving contraction of F. Similarly, the inequality holds for all $\tau\geq\tau_2(\varepsilon)$ because G^ε is weakly higher than F above $\tau_2(\varepsilon)$. For $[\tau_1(\varepsilon),\tau_2(\varepsilon)]$, recall from Lemma 12 that there exist $\delta,\sigma>0$ such that $\delta+\int_{\underline{\theta}}^{\tau}G(s)ds\leq\int_{\underline{\theta}}^{\tau}F(s)ds$ for any $\tau\in[\tau_1-\sigma,\tau_2]$. Now choose ε small enough so that $\int_{\underline{\theta}}^{\tau}G^\varepsilon(s)ds\leq\delta+\int_{\underline{\theta}}^{\tau}G(s)ds$ for any $\tau\in[\tau_1-\sigma,\tau_2]$ and further that $\tau_1-\sigma\leq\tau_1(\varepsilon)$. The two inequalities then imply that $\int_{\underline{\theta}}^{\tau}G^\varepsilon(s)ds\leq\delta+\int_{\underline{\theta}}^{\tau}G(s)ds\leq\int_{\underline{\theta}}^{\tau}F(s)ds$ for any $\tau\in[\tau_1(\varepsilon),\tau_2(\varepsilon)]$.

To complete the proof, we show that for small enough $\varepsilon>0$, the revenue guarantee of $(G^\varepsilon,\phi(\varepsilon))$ is strictly higher than that of (G,ϕ). First notice that $\tau_1(\varepsilon)$ is a weak-highest equilibrium threshold for the test-fee structure $(G^\varepsilon,\phi(\varepsilon))$. This is because G^ε is exponential from $\tau_1(\varepsilon)$ to $\tau_2(\varepsilon)$ and flat from $\tau_2(\varepsilon)$ to $\mu+\phi_d(\varepsilon)$. The intermediary's revenue is

$$\begin{aligned}
R(\varepsilon)&\equiv\phi_t(\varepsilon)+\phi_d(\varepsilon)\big(1-G^\varepsilon(\tau_1(\varepsilon))\big)\\
&=\int_{\mu+\phi_d(\varepsilon)}^{\overline{\theta}}[s-\mu-\phi_d(\varepsilon)]dG^\varepsilon(s)+\phi_d(\varepsilon)\big(1-G^\varepsilon(\tau_1(\varepsilon))\big)\\
&=\overline{\theta}-\mu-\phi_d(\varepsilon)-\int_{\mu+\phi_d(\varepsilon)}^{\overline{\theta}}G^\varepsilon(s)ds+\phi_d(\varepsilon)-\int_{\underline{\theta}}^{\tau_1(\varepsilon)}G^\varepsilon(s)ds\\
&=\int_{\tau_1(\varepsilon)}^{\mu+\phi_d(\varepsilon)}G^\varepsilon(s)ds\\
&=\kappa e^{\frac{\tau_2(\varepsilon)-\tau_1(\varepsilon)}{\tau_1(\varepsilon)-\tau_0}}\big(2(\tau_1(\varepsilon)-\tau_0)+\mu-\tau_2(\varepsilon)\big)-(\tau_1(\varepsilon)-\tau_0)\kappa.
\end{aligned}$$

To show that there exists $\varepsilon>0$ with $R(\varepsilon)>R(0)$, we consider the derivative of revenue with respect to ε, $R'(\varepsilon)$. This derivative exists because $\tau_1'(\varepsilon) = -1$, and $\tau_2'(\varepsilon)$ exists and $|\tau_2'(\varepsilon)| < -\infty$ as argued above. Now let $\lambda(\varepsilon) = \frac{\tau_2(\varepsilon)-\tau_1(\varepsilon)}{\tau_1(\varepsilon)-\tau_0}$ and write

$$R'(\varepsilon) = \kappa e^{\lambda(\varepsilon)}\Big(\lambda'(\varepsilon)\big(2(\tau_1(\varepsilon)-\tau_0)+\mu-\tau_2(\varepsilon)\big) - 2 - \tau_2'(\varepsilon)\Big) + \kappa.$$

We now evaluate this derivative at $\varepsilon = 0$. The fact that G is a mean-preserving spread of F implies that $\kappa = e^{-\frac{\mu-\tau_0}{\tau_1-\tau_0}}$ and therefore $\kappa e^{\lambda(0)} = 1$. Also $2(\tau_1 - \tau_0) + \mu - \tau_2 = \phi_d$. So we have

$$\begin{aligned} R'(0) &= \lambda'(0)\phi_d - 2 - \tau_2'(0) + \kappa \\ &= \tau_2'(0) + 1 - \frac{\tau_2-\tau_1}{\tau_1-\tau_0} - 2 - \tau_2'(0) + \kappa \\ &= \kappa - \frac{\tau_2-\tau_1}{\tau_1-\tau_0} - 1 \\ &= e^{\frac{\tau_0-\mu}{\tau_1-\tau_0}} + \frac{\tau_0-\mu}{\tau_1-\tau_0} - 1 \\ &> 0. \end{aligned}$$

The inequality follows because $e^x > 1 + x$ for $x \neq 0$. Thus $R(\varepsilon) > 0$ for small enough ε. This contradicts the optimality of (G, ϕ). □

B.3 Proof of Proposition 5 on p. 26

Proof. Consider a test-fee structure with a zero disclosure fee. By (w-P), the revenue is

$$\phi_t \leq \int_\mu^{\overline{\theta}} (s-\mu)dG(s) = (\overline{\theta} - \mu) - \int_\mu^{\overline{\theta}} G(s)ds.$$

This revenue is maximized when the inequality holds as an equality and $\int_\mu^{\overline{\theta}} G(s)ds$ is minimized. Since $\int_\mu^{\overline{\theta}} G(s)ds \geq \int_\mu^{\overline{\theta}} F(s)ds$ for any G that is a mean-preserving contraction of F, $\int_\mu^{\overline{\theta}} G(s)ds$ is minimized when $\int_\mu^{\overline{\theta}} G(s)ds = \int_\mu^{\overline{\theta}} F(s)ds$. So the robustly optimal revenue is $\int_\mu^{\overline{\theta}}(s-\mu)dF(s)$, which can be achieved by a binary test G with $s_H = E[\theta|\theta \geq \mu]$, $s_L = E[\theta|\theta < \mu]$ and $G(s_L) = \Pr[\theta < \mu]$. To see this, first note that $G \in \Gamma(F)$ because G is induced by a test that maps $\theta < \mu$ to s_L and $\theta \geq \mu$ to s_H. Second, the revenue is $\int_\mu^{\overline{\theta}}(s-\mu)dG(s) = (1-G(s_L))(s_H - \mu) = \Pr(\theta \geq \mu)\,(E[\theta|\theta \geq \mu] - \mu) = \int_\mu^{\overline{\theta}}(s-\mu)dF(s)$. □

B.4 Proof of Proposition 6 on p. 27

Proof. Suppose $G \in \Gamma(F)$ is restricted to have binary support $s_1, s_2 \in [\underline{\theta}, \overline{\theta}]$. Without loss of generality, let $s_1 < s_2$. Notice that for any s_1, s_2, the mean-preserving contraction constraints pins down $G(s_1) = \frac{s_2-\mu}{s_2-s_1}$. From arguments similar to those used in proving Lemma 2, we focus on $\phi_t, \phi_d \geq 0$. To satisfy (w-P), $\phi_d \leq s_2 - \mu$. For any $\phi_d \in [0, s_2 - \mu]$, (w-HE) implies that the lowest weak-highest equilibrium threshold is $\tau = s_1 + \phi_d \in [s_1, s_1 + s_2 - \mu) \subset [s_1, s_2)$, so $G(\tau) = G(s_1)$.

Thus, the intermediary's problem can be written as

$$\max_{\phi_t, \phi_d, s_1, s_2} \phi_t + \phi_d (1 - G(s_1)) \tag{19}$$

$$s.t. \quad \phi_t \leq \int_{\mu+\phi_d}^{\bar{\theta}} [s - (\mu + \phi_d)] dG(s)$$

$$\phi_d \in [0, s_2 - \mu]$$

$$\underline{\theta} \leq s_1 < \mu < s_2 \leq \overline{\theta}$$

$$G \in \Gamma(F).$$

Clearly the constraint for ϕ_t must bind, so $\phi_t = \int_{\mu+\phi_d}^{\bar{\theta}} [s - (\mu + \phi_d)] dG(s) = (s_2 - \mu - \phi_d)(1 - G(s_1))$. Plugging in ϕ_t, we have $\phi_t + \phi_d(1 - G(s_1)) = (s_2 - \mu)(1 - G(s_1))$.

The problem can be further simplified to

$$\max_{\phi_d, s_1, s_2} (s_2 - \mu)(1 - G(s_1))$$

$$s.t. \quad \phi_d \in [0, s_2 - \mu]$$

$$\underline{\theta} \leq s_1 < \mu < s_2 \leq \overline{\theta}$$

$$G \in \Gamma(F).$$

Since ϕ_d does not enter the objective function, for any optimal test, any $\phi_d \in [0, s_2 - \mu]$ and $\phi_t = (s_2 - \mu - \phi_d)(1 - G(s_1))$ form an optimal fee structure. In particular, $\phi_d = 0$ and $\phi_t = (s_2 - \mu)(1 - G(s_1))$ are optimal. Recall that from Proposition 5, a binary test $s_1 = E[\theta|\theta < \mu]$, $s_2 = E[\theta|\theta \geq \mu]$ is optimal when the intermediary is restricted to using only a testing fee. So $s_1 = s_L = E[\theta|\theta < \mu]$, $s_2 = s_H = E[\theta|\theta \geq \mu]$ is also an optimal test for problem (19). □

B.5 Proof of Proposition 8 on p. 28

Proof. Consider an evidence-test-fee structure denoted by fees (ϕ_t, ϕ_d), an unbiased test $T : \Theta \to \Delta S$, and an evidence structure $M : S \rightrightarrows \mathcal{M}$ such that for each s, $M(s)$ is a Borel space. A strategy profile (σ, p) consists of the agent's strategy $\sigma = (\sigma_T, \sigma_D)$, where $\sigma_T \in [0, 1]$ and σ_D maps $s \in S$ to $\Delta(M(s) \cup \{N\})$, and the market price $p : \mathcal{M} \to [\underline{\theta}, \bar{\theta}]$. Let (σ, p) be an adversarial equilibrium. We first consider the case in which the agent has the asset tested with probability 1, that is $\sigma_T = 1$.

Consider the disclosure stage. Let $G_{(\sigma,p)}$ be the induced distribution of prices, i.e., $G_{(\sigma,p)}(x) = Pr[p(\sigma_D(s)) \leq x]$ for any x, taking into account both the randomization over the score and the agent's strategy. Let $\tau \equiv p(N) + \phi_d$. We show that the following holds, mirroring our characterization of the highest equilibrium threshold (HE):

$$\tau - \phi_d \leq E_{G_{(\sigma,p)}}[x|x \leq \tau], \tag{20}$$

$$\tau' - \phi_d > E_{G_{(\sigma,p)}}[x|x \leq \tau'], \forall \tau' > \tau. \tag{21}$$

Let us argue why (20) holds. Since $\tau - \phi_d = p(N)$, it suffices to show that $p(N)$ is weakly less than $E_{G_{(\sigma,p)}}[x|x \leq \tau]$. Observe that with probability 1, $p(\sigma_D(s))$ is at least $p(N)$: if $p(\sigma_D(s))$ were strictly less than $p(N)$, the agent could profitably deviate to sending message N and obtaining a strictly higher price. But this implies that $p(N) \leq E[p(\sigma_D(s))|p(\sigma_D(s)) \leq \tau] = E_{G_{(\sigma,p)}}[x|x \leq \tau]$.

To see why (21) holds, suppose for contradiction that $\tau'' - \phi_d \leq E_{G_{(\sigma,p)}}[p|p \leq \tau'']$ for some $\tau'' > \tau$. By Lemma 5, there exists $\tau' > \tau$ such that $\tau' - \phi_d = E_{G_{(\sigma,p)}}[p|p \leq \tau']$. Consider the strategy profile (σ', p') defined as follows. The agent's strategy σ' is the same as σ except that the agent conceals a score s if $p(\sigma(s)) \leq \tau'$. For $m \neq N$, $p'(m) = p(m)$, and for $m = N$, the price is $p'(m) = E_{G_{(\sigma,p)}}[x|x \leq \tau'] \geq E_{G_{(\sigma,p)}}[x|x \leq \tau] \geq p(m)$. Notice that since any message m that is disclosed in (σ', p') is also disclosed in (σ, p), the prices p' are defined on path via Bayes rule.

To see that (σ', p') is an equilibrium, consider any score s such that $p(\sigma(s)) > \tau$ with positive probability. Therefore, following a score of s, the agent optimally randomizes over messages other than N that lead to the same (and maximal) price which, abusing notation, we denote by $p(\sigma(s))$. Since $p(m) = p'(m)$ for all $m \neq N$, $\sigma(s)$ is optimal among all strategies that send N with probability 0 given prices p'. Therefore, for such a score it is optimal to follow $\sigma(s)$ if $p'(\sigma(s)) > \tau' = p'(N) + \phi_d$, and to conceal if $p'(\sigma(s)) \leq \tau'$, as prescribed by σ'. Now consider a score s such that $p(\sigma(s)) \leq \tau$ with probability 1. For such a score, it is optimal given prices p to conceal, i.e., for any message $m \neq N$ that the agent can send following a score of s, $p(m) - \phi_d \leq p(N)$. Since $p'(m) = p(m)$ and $p'(N) \geq p(N)$, it is also optimal to conceal given prices p', as prescribed by σ'.

Now consider the testing stage. If

$$\phi_t \geq \int_{\mu+\phi_d}^{\overline{\theta}} [x - (\mu + \phi_d)] dG_{(\sigma,p)}, \tag{22}$$

then there exists an equilibrium in the evidence-test-fee structure in which the agent has the asset tested with probability 0. The argument parallels that of Lemma 1. In particular, consider a strategy profile (σ', p') such that that $\sigma'_T = 0$, off-path the agent follows $\sigma(s)$ if $p'(\sigma(s)) > \mu + \phi_d$ ($p'(\sigma(s))$ is well-defined as argued above) and otherwise conceals, and the prices are $p'(N) = \mu$ and $p(m) = p'(m)$ for all $m \neq N$. Since the set of disclosed messages in (σ', p') is a subset of that in (σ, p), an argument similar to above shows that the agent's disclosure strategy is sequentially rational. Also, by deviating to taking the test, the agent receives an expected payoff lower than μ,

$$\int_{\underline{\theta}}^{\mu+\phi_d} \mu dG_{(\sigma,p)} + \int_{\mu+\phi_d}^{\bar{\theta}} [x - \phi_d] dG_{(\sigma,p)} - \phi_t = \mu + \int_{\mu+\phi_d}^{\bar{\theta}} [x - (\mu + \phi_d)] dG_{(\sigma,p)} - \phi_t \leq \mu,$$

where the inequality follows from (22). Therefore, the revenue in an adversarial equilibrium is at most zero, which is obtained by any test-fee structure with zero fees. So suppose that (22) is violated and consider a test-fee structure $(G_{(\sigma,p)}, \phi)$.

By Lemma 3, τ is a weak-highest equilibrium threshold of the test-fee structure. Also, since (22) is violated, by Lemma 1 the test is taken with probability 1 in all equilibria. Therefore, the revenue in an adversarial equilibrium of this test-fee structure is equal to the revenue in an adversarial equilibrium of the evidence-test-fee environment.

We now consider the case $\sigma_T \in [0, 1)$. Notice that in this case

$$\phi_t \geq \int_{p(N)+\phi_d}^{\bar{\theta}} [x - (p(N) + \phi_d)] dG_{(\sigma,p)} \geq \int_{\mu+\phi_d}^{\bar{\theta}} [x - (\mu + \phi_d)] dG_{(\sigma,p)},$$

so (22) holds. The argument above shows that the revenue in an adversarial equilibrium is at most zero, which can be obtained in a test-fee structure with any test and zero fees. ☐

B.6 Using Only Disclosure Fees

Proposition 9. *Suppose that the intermediary is restricted to using only disclosure fees. Then there exists a test-fee structure (G, ϕ) in the step-exponential class, where $G(\tau_2) = 1$, that is robustly optimal.*

Proof. Consider any optimal test-fee structure (G', ϕ') when the intermediary is restricted to a disclosure fee of zero (an optimal test-fee structure exists by an argument similar to Lemma 4). Lemma 11 shows that that there exists a test-fee structure (G, ϕ) with $\phi_t \leq \phi'_t$ in the step-exponential-step class that generates a weakly higher revenue. Since $\phi_t \leq \phi'_t$, the intermediary receives a weakly higher revenue from the disclosure fee in (G, ϕ) than in (G', ϕ'), so (G, ϕ) also maximizes revenue when the intermediary is restricted to a testing fee of zero.

Suppose that $G(\tau_2) < 1$ and that (G, ϕ) is parameterized by $\kappa, \tau_0, \ldots, \tau_3$ as defined in (5), (6), and (7). The robust revenue of this test-fee structure from the disclosure fee is $\phi_d(1 - \kappa) =$

$(\tau_1 - \tau_0)(1-\kappa)$. Note that $\tau_2 < \tau_3$ since $G(\tau_2) < 1 = G(\tau_3)$. For $\varepsilon_1 \in [0,1]$ and $\varepsilon_2 \in [0, \tau_3 - \tau_2]$, consider $(G_{\varepsilon_1,\varepsilon_2}, \phi)$ parameterized by $\kappa', \tau_0', \dots, \tau_3'$ such that $\kappa' = \kappa - \varepsilon_1$, $\tau_2' = \tau_2 + \varepsilon_2$, and $\tau_i' = \tau_i$ for $i = 0, 1, 3$. We show that there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $G_{\varepsilon_1,\varepsilon_2}$ is a mean-preserving contraction of G and gives a strictly higher revenue from a disclosure fee of $(\tau_1 - \tau_0)(1 - \kappa + \varepsilon_2)$ and therefore is robustly optimal.

For small enough $\varepsilon_1, \varepsilon_2$ so that $G_{\varepsilon_1,\varepsilon_2}(\tau_2') \leq 1$, $G_{\varepsilon_1,\varepsilon_2}$ is a well defined distribution function. Notice that $G_{\varepsilon_1,\varepsilon_2}$ is decreasing in ε_1, and $\int_{\underline{\theta}}^{\overline{\theta}} G_{0,\varepsilon_2}(s)ds - \int_{\underline{\theta}}^{\overline{\theta}} G_{\varepsilon_1,\varepsilon_2}(s)ds \geq \int_{\underline{\theta}}^{\tau_1} G_{0,\varepsilon_2}(s)ds - \int_{\underline{\theta}}^{\tau_1} G_{\varepsilon_1,\varepsilon_2}(s)ds = \varepsilon_1(\tau_1 - \tau_0) \geq 0$. Since $\int_{\underline{\theta}}^{\overline{\theta}} G_{0,\varepsilon_2}(s)ds - \int_{\underline{\theta}}^{\overline{\theta}} G(s)ds$ is continuous in ε_2 and goes to 0 as ε_2 goes to 0, for small enough $\varepsilon_2 > 0$ there exists $\varepsilon_1 > 0$ such that $\int_{\underline{\theta}}^{\overline{\theta}} G_{\varepsilon_1,\varepsilon_2}(s)ds = \int_{\underline{\theta}}^{\overline{\theta}} G(s)ds$. Moreover, $G_{\varepsilon_1,\varepsilon_2}(s) \leq G(s)$ for $s \leq s^*$, and $G_{\varepsilon_1,\varepsilon_2}(s) \geq G(s)$ for $s \geq s^*$, where $s^* = \tau_2 - (\tau_1 - \tau_0)\log\alpha$ is the unique intersection of $G_{\varepsilon_1,\varepsilon_2}$ and G on the interval $(\tau_2, \tau_2 + \varepsilon_2)$. Thus, $G_{\varepsilon_1,\varepsilon_2}$ is a mean-preserving contraction of G. $\square$